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VARIOUS WEAKER FORMS OF COMMUTING MAPPINGS IN FUZZY METRIC SPACE

NANDU PRASAD KOIRI, RAJEEV KUMAR, K. B. MANANDHAR, AND AJAY KUMAR CHAUDHARY*

Abstract. In 1968, K. Goebel [21] did research on more than one self-mapping and established the coincidence theorem in order to derive the common fixed point theorem. Despite the fact that Machuca [35] initially examined this issue in 1967 under some strict topological restrictions, in 1976 Jungck [10] introduced commuting mappings and established common fixed point theorems by using constructive procedures of iterative sequences. This condition is too strong. Continuing this, Sessa [44] gave weakly commuting mappings in 1982 and extended a variety of fixed point theorems by substituting weakly commutativity for commutativity. Owing to the elegance of this result, many authors have introduced various weaker forms of more than one self-mapping in metric and fuzzy metric spaces.

In this paper, we discuss various non-commuting mappings in metric and fuzzy metric spaces. This article updates the comparative study on non-commuting maps and shows their interrelationship in these spaces.

Key Words: fuzzy metric space, commuting mapping, weakly commuting mapping, compatible mappings

Mathematics Subject Classification: 47H09, 47H10, 46S40

1. Introduction

In 1965, fuzzy sets were proposed by L. A. Zadeh [23] as a mechanism to capture the ambiguity of everyday life and provided the foundation for the development of fuzzy mathematics. In 1975, Kramosil and Michalek [19] introduced the notion of fuzzy metric space by generalizing the concept of M. Frchet metric space [26]. Heilpern [40] first introduced the concept of fuzzy contractive mappings in 1981, and M. Grabiec [27] proved the contraction principle in fuzzy metric spaces in 1988. In 1994, George and Veeramani [1] modified the concept of fuzzy metric space with the help of t-norms. Afterwards, many researchers have done work in this domain and established fixed point results in fuzzy metric spaces.

Stephen Banachs contraction mapping [39] works with a single self-mapping and obtains the fixed point theorem. A common fixed point of f with the identity mapping on X can be thought of as the study of fixed points of self-mappings. However, K. Goebel [21] introduced in 1968 the notion of substituting another self-mapping g on X for the identity mapping and established the coincidence theorem in order to derive the common fixed point theorem. Despite the fact that Machuca [35] initially examined this issue in 1967 under some strict topological restrictions, G. Jungck [10] introduced the notion of commuting mappings in 1976, and S. Sessa [44] generalized this concept by proposing weakly commuting mappings in 1982. In 1986, G. Jungck [11] introduced the notion of compatible mappings, which is more general than commuting and weakly commuting mappings.

Due to the elegance of this result, many authors have introduced various other contractive conditions on more than one self-mapping, such as compatible-type conditions. Readers may see references [2, 4, 5, 6, 7, 14, 24, 25, 31, 33, 34, 36, 42, 43, 45].

This paper presents various non-commuting and compatible mappings in metric and fuzzy metric spaces, which helps in comparative and interrelationship studies in these spaces.

2. Preliminaries

Definition 2.1 (Maurice Frchet, 1906). Let M be a non-empty set and $d : M \times M \to \mathbb{R}$ a real-valued function satisfying for all $x, y, z \in M$:

(i) $d(x,y) \geq 0$ [Positivity]
(ii) $d(x,y) = 0 \Leftrightarrow x = y$ [Reflexivity]
(iii) $d(x,y) = d(y,x)$ [Symmetry]
(iv) $d(x,y) \leq d(x,z) + d(z,y)$ [Triangle inequality]

Then (M,d) is called a **metric space**.

Definition 2.2 [Karl Menger -1942]. A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a **triangular norm** (shortly **T-norm**) if it satisfies the following conditions:

For all $a, b, c, d \in [0,1]$:

(1) $T(a,1) = a$ [Neutral Element]
(2) $T(a,b) = T(b,a)$ [Commutativity]
(3) $T(a, T(b,c)) = T(T(a,b), c)$ [Associativity]
(4) $T(a,b) \leq T(c,d)$, whenever $a \leq c$ and $b \leq d$ [Monotonicity]

Example 2.3 [3]: [A. K. Chaudhary et al.,-2021]

$$T(a,b) = \begin{cases} 0, & \text{for } a = b = 0, \\ a \text{ or } b, & \text{for } b = 1 \text{ or } a = 1, \\ \dfrac{ab}{a+b}, & \text{otherwise.} \end{cases}$$

Then T is a triangular norm.

Definition 2.4 [8]: [B. Schweizer and A. Sklar- 1960]

A triangular norm T is said to be a **continuous T-norm** if it satisfies the following conditions: For each $a, b, c, d \in [0,1]$:

(1) T is associative and commutative;
(2) T is continuous;
(3) **Boundary condition:** $T(a,1) = a$;
(4) **Monotonicity:** $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$.

Example 2.5 $T(a,b) = ab$ for $a, b \in [0,1]$ is a continuous t-norm.

Definition 2.6[23] [Zadeh, 1965] If X is a universal set and $x \in X$, then a fuzzy set A defined on X is a collection of ordered pairs:

$$A = \{(x, \mu_A(x)) : x \in X, \mu_A(x) \in [0,1]\}$$

where $\mu_A : X \to [0,1]$ is a membership function.

Example 2.7 Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and A is a fuzzy set of smart students.

$$A = \{(x_1, 0.2), (x_2, 0), (x_3, 0.3), (x_4, 0.5), (x_5, 0.9), (x_6, 0.1)\}$$

Where, $\mu_A(x) \in [0,1]$, i.e., the degree of smartness of x_1 is 0.2 and so on.

Definition 2.8 [19]: [I. Kramosil and J. Michalek, 1975]

The 3-tuple $(X, M, *)$ is said to be a **Fuzzy metric space** if X is an arbitrary set, $*$ is a continuous T-norm, and $M : X \times X \times [0, \infty) \to [0, 1]$ is a fuzzy set satisfying the following conditions:

For all $x, y, z \in X$ and $t, s > 0$:

(1) $M(x, y, 0) = 0$;
(2) $M(x, y, t) = 1 \iff x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(5) $M(x, y, \cdot) : [0, \infty) \to [0, 1]$ is left continuous.

It is also called a **KM-Fuzzy metric space**.

Definition 2.9 [1]: [A. George and P. Veeramani, 1994]

The 3-tuple $(X, M, *)$ is said to be a **Fuzzy metric space (GV FM)** if X is an arbitrary set, $*$ is a continuous T-norm, and $M : X \times X \times [0, \infty) \to [0, 1]$ is a fuzzy set satisfying the following conditions:

For all $x, y, z \in X$ and $t, s > 0$:

(1) $M(x, y, t) > 0$;
(2) $M(x, y, t) = 1 \iff x = y$;
(3) $M(x, y, t) = M(y, x, t)$;
(4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$;
(5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ is continuous.

Example 2.10 Let (X, d) be a metric space. Define $a * b = ab$ for $a, b \in [0, 1]$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}, \quad k, m, n \in \mathbb{R}$$

Then $(X, M, *)$ is a fuzzy metric space induced by d. If $k = m = n = 1$,then the above equation reduces to $M(x, y, t) = \frac{t}{t+d(x,y)}$ is called the **standard fuzzy metric**.

3. Non-commuting mappings in metric Space

Definition 3.1[9]:

If S and T are two self-mappings of a metric space (X, d), then S and T are called **commuting mappings** on X if

$$d(STx, TSx) = 0 \quad \text{for all } x \in X.$$

Definition 3.2 [9]:

If S and T are two self-mappings of a metric space (X, d), then S and T are called **weakly commuting mappings** on X if

$$d(STx, TSx) \leq d(Sx, Tx) \quad \text{for all } x \in X.$$

Remark 3.3: Every commuting mapping is weakly commuting, but the converse is not true.

Example 3.4[44]: Let $X = [0, 1]$ be equipped with the usual metric d on X. Define constant mappings $S, T : X \to X$ by

$$Sx = a \quad \text{and} \quad Tx = b, \qquad a \neq b.$$

Then S and T are weakly commuting but not commuting, since

$$d(STx, TSx) = |a - b| = d(Sx, Tx).$$

Definition 3.5 [11]: [G. Jungck et al., 1986]

Two self-mappings S and T of a metric space (X, d) are said to be **compatible mappings** or **asymptotically commuting mappings** if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Definition 3.6 [11]:
Two self-mappings S and T of a metric space (X, d) are said to be **compatible mappings of type (A)** if

$$\lim_{n\to\infty} d(STx_n, TTx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(TSx_n, SSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Remark 3.7: Compatible mappings and compatible mappings of type (A) are independent of each other.
Example 3.8[15]:
Let $X = \mathbb{R}$ equipped with the usual metric d. Define self-mappings S and T as follows:

$$Sx = x, \qquad Tx = \begin{cases} 0, & x \in \mathbb{Z}, \\ 1, & x \notin \mathbb{Z}. \end{cases}$$

Consider the sequence $\{x_n\}$ given by $x_n = 1 + \frac{1}{n+1}$, $n > 0$.
Example 3.9[37]:
Let $X = [2, 12]$ and d be the usual metric on X. Define $S, T : X \to X$ as follows:

$$Sx = \begin{cases} 2, & x = 2 \text{ or } x > 5, \\ 12, & 2 < x \leq 5, \end{cases} \qquad Tx = \begin{cases} 2, & x = 2, \\ 12, & 2 < x \leq 5, \\ \dfrac{x+1}{3}, & x > 5. \end{cases}$$

Consider the sequence $\{x_n\}$ given by

$$x_n = 5 + \frac{1}{n}, \quad n > 0.$$

Then S and T are **compatible mappings of type (A)**, but not compatible mappings.
Then S and T are compatible but not compatible of type (A).
Definition 3.10 [16]: [H. K. Pathak and M. S. Khan, 1995]
Two self-mappings S and T of a metric space (X, d) are said to be **compatible mappings of type (B)** if

$$\lim_{n\to\infty} d(STx_n, TTx_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, SSx_n)\Big],$$

and

$$\lim_{n\to\infty} d(TSx_n, SSx_n) \leq \frac{1}{2}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, TTx_n)\Big],$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Definition 3.11[17]: [H. K. Pathak et al., 1996]

Two self-mappings S and T of a metric space (X, d) are said to be **compatible mappings of type (P)** if

$$\lim_{n\to\infty} d(SSx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Example 3.12:
Let $X = [0, 3]$ with the usual metric $d(x, y) = |x - y|$. Define self-mappings S and T on X as follows:

$$Sx = Tx = 2, \quad \text{for } x \in [0, 2), \qquad Sx = Tx = \frac{5}{3}, \quad \text{for } x = 2,$$

$$Sx = 4 - x, \qquad Tx = x, \quad \text{for } x \in (2, 3].$$

Then S and T are **compatible mappings of type (A) and type (P)**.
Definition 3.13 [18]: [H. K. Pathak et al., 1998]
Two self-mappings S and T of a metric space (X, d) are said to be **compatible mappings of type (C)** if

$$\lim_{n\to\infty} d(STx_n, TTx_n) \le \frac{1}{3}\Big[\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, SSx_n) + \lim_{n\to\infty} d(St, TTx_n)\Big],$$

and

$$\lim_{n\to\infty} d(TSx_n, SSx_n) \le \frac{1}{3}\Big[\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, TTx_n) + \lim_{n\to\infty} d(Tt, SSx_n)\Big],$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Remark 3.14: Compatible mappings of type (A) $\Longrightarrow$ Compatible mappings of type (B) $\Longrightarrow$ Compatible mappings of type (C), but the converse is not true in general.
Example 3.15[13]:
Let $X = [2, 12]$ with the usual metric d. Define $S, T : X \to X$ as follows:

$$Sx = \begin{cases} 1, & x = 1, \\ 3, & 1 < x \le 7, \end{cases} \qquad Tx = \begin{cases} 1, & x \in \{1\} \cup (7, 20], \\ 2, & 1 < x \le 7, \\ x - 6, & 7 < x \le 20. \end{cases}$$

Consider the sequence $\{x_n\}$ given by

$$x_n = 7 + \frac{1}{n}, \quad n > 0.$$

Then S and T are **compatible of type (C)** but neither compatible, nor compatible of type (A) or type (B).
Definition 3.16[29]: [M. R. Singh and Y. Mahendra Singh, 2007]
Two self-mappings S and T of a metric space (X, d) are said to be **compatible mappings of type (E)** if

$$\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} TSx_n = St \quad \text{and} \quad \lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Tt,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Remarks 3.17:

- If $St = Tt$, then compatible mappings of type (E) imply compatible mappings of type (A), (B), (C), and (P), but the converse may not be true.
- If $St \neq Tt$, then compatible mappings of type (E) are neither compatible mappings, nor compatible mappings of type (A), (C), or (P).

Example 3.18[29]:
Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define self-mappings S and T as:

$$Sx = Tx = \frac{1}{2}, for \quad x \in \left[0, \frac{1}{2}\right), \qquad Sx = Tx = \frac{2}{3}, for \quad x = \frac{1}{2},$$
$$Sx = 1 - x, \quad Tx = x, for \quad x \in \left(\frac{1}{2}, 1\right].$$

Consider a sequence $\{x_n\} \subset X$ such that $x_n \to \frac{1}{2}$ and $x_n > \frac{1}{2}$ for all n. Then $St = Tt = \frac{2}{3}$, so S and T are compatible [compatible of type (A), (B), (C), (P)], but not of type (E).

Example 3.19 [29]:
Let $X = [0, 1]$ with the usual metric $d(x, y) = |x - y|$. Define S and T as:

$$Sx = 1, \quad Tx = 0, for \quad x \in \left[0, \frac{1}{2}\right] - \left\{\frac{1}{4}\right\}, \qquad Sx = 0, \quad Tx = 1, for \quad x = \frac{1}{4},$$
$$Sx = \frac{1-x}{4}, \quad Tx = \frac{x}{2}, for \quad x \in \left(\frac{1}{2}, 1\right].$$

Consider a sequence $\{x_n\} \subset X$ such that $x_n \to \frac{1}{2}$ and $x_n > \frac{1}{2}$ for all n. Then $St = 0 \neq 1 = Tt$, so S and T are compatible of type (E), but neither compatible nor compatible of type (A), (C), or (P).

Definition 3.20[28]: [M. R. Singh and Y. Mahendra Singh, 2011]
Two self-mappings S and T of a metric space (X, d) are said to be **T-compatible of type (E)** if

$$\lim_{n\to\infty} TTx_n = \lim_{n\to\infty} TSx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Definition 3.21 [28]:
Two self-mappings S and T of a metric space (X, d) are said to be **S-compatible of type (E)** if

$$\lim_{n\to\infty} SSx_n = \lim_{n\to\infty} STx_n = Tt,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Remark 3.22: Compatible mappings of type (E) imply both S- and T-compatible mappings of type (E); however, S- or T-compatible of type (E) do not necessarily imply compatible mappings of type (E).

Example 3.23 [28]:

Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$. Define self-mappings S and T as:

$$Sx = 1, \quad Tx = \frac{1}{5}, for \quad x \in \left[0, \frac{1}{2}\right] - \left\{\frac{1}{4}\right\}, \qquad Sx = 0, \quad Tx = 1, for \quad x = \frac{1}{4},$$

$$Sx = \frac{1-x}{4}, \quad Tx = \frac{x}{2}, for \quad x \in \left(\frac{1}{2}, 1\right].$$

Consider a sequence $\{x_n\} \subset X$ such that $x_n \to \frac{1}{2}$ and $x_n > \frac{1}{2}$ for all n. Then S and T are **S-compatible of type (E)** but not compatible of type (E).

Definition 3.24[47]: [Y. Rohen and M. R. Singh, 2008]

Two self-mappings S and T of a metric space (X,d) are said to be **compatible mappings of type (R)** if

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(SSx_n, TTx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Definition 3.25[22]: [K. Jha, V. Popa, and K. B. Manandhar, 2014]

Two self-mappings S and T of a metric space (X,d) are said to be **compatible mappings of type (K)** if

$$\lim_{n\to\infty} SSx_n = Tt \quad \text{and} \quad \lim_{n\to\infty} TTx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad \text{for some } t \in X.$$

Remark 3.26: Compatible mappings of type (K) are independent of compatible mappings of type (B), (C), and (P).

Example 3.27 [20]:

Let $X = [0,2]$ with the usual metric $d(x,y) = |x-y|$. Define self-mappings S and T as:

$$Sx = 2, \quad Tx = 0, for \quad x \in [0,1] - \left\{\frac{1}{2}\right\}, \qquad Sx = 0, \quad Tx = 2, for \quad x = \frac{1}{2},$$

$$Sx = \frac{2-x}{2}, \quad Tx = \frac{x}{2}, for \quad x \in (1,2].$$

Consider a sequence $\{x_n\} \subset X$ such that

$$x_n = 1 + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Then S and T are compatible of type (K) and (E), but neither compatible nor compatible of type (B), (C), or (P).

4. Non-commuting mappings in a fuzzy metric Space:

Definition 4.1[41]: [S. N. Mishra et al., 1994]

The self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **compatible** if

$$\lim_{n\to\infty} M(STx_n, TSx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $x \in X$ and $t > 0$.

Definition 4.2 [46]: [Y. J. Cho et al., 1998]

The self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **compatible of type (A)** if

$$\lim_{n\to\infty} M(STx_n, TTx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} M(TSx_n, SSx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$$

for some $t \in X$ and $t > 0$.

Definition 4.3 [46]:

The self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **compatible of type (P)** if

$$\lim_{n\to\infty} M(TTx_n, SSx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

Definition 4.4 [38]: [R. Vasuki, 1999]

Two self-maps S and T of a fuzzy metric space $(X, M, *)$ are said to be **weakly commuting** if

$$M(STx, TSx, t) \geq M(Sx, Tx, t) \quad \text{for every } x \in X.$$

Definition 4.5 [30]:

Two self-maps S and T of a fuzzy metric space $(X, M, *)$ are said to be **R-weakly commuting** provided there exists a positive real number R such that

$$M(STx, TSx, t) \geq M(Sx, Tx, \frac{t}{R}) \quad \text{for all } x \in X.$$

Remark 4.6: Weak commutativity implies R-weak commutativity, and the converse is true for $R \leq 1$.

Definition 4.7[9]: [B. Singh and S. Jain, 2005]

Two self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **semi-compatible** if

$$M(STx_n, Tx_n, t) \to 1 \quad \text{for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$Sx_n, Tx_n \to p \quad \text{for some } p \in X \text{ as } n \to \infty.$$

It follows that S and T are semi-compatible and $Sy = Ty \Rightarrow STy = TSy$ by taking $\{x_n\} = y$ and $x = Sy = Ty$.

Remark 4.8: For two self-mappings S and T of a fuzzy metric space $(X, M, *)$, R-weakly commuting implies semi-compatible, but the converse is not true.

Example 4.9 [30]:

Let $X = [0, 2]$ and $*$ be defined by $a * b = \min\{a, b\}$. Let

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

be the standard fuzzy metric induced by d, where $d(x, y) = |x-y|$ for all $x, y \in X$. Define self-mappings S and T as:

$$Sx = \begin{cases} 2, & x \in [0, 1], \\ x, & x \in (1, 2], \end{cases} \qquad Tx = \begin{cases} 1, & x \in [0, 1), \\ x + \frac{3}{5}, & x \in (1, 2]. \end{cases}$$

Then S and T are semi-compatible but not R-weakly commuting mappings.

Definition 4.10[9]:

The self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **weakly compatible** or **coincidentally commuting** if S and T commute at their coincidence points, i.e., for $x \in X$, if $Sx = Tx$, then

$$STx = TSx.$$

Definition 4.11[29]: [M. R. Singh and Y. M. Singh, 2007]

The self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **compatible of type (E)** if

$$\lim_{n\to\infty} M(SSx_n, STx_n, t) = 1, \quad \lim_{n\to\infty} M(SSx_n, Tx, t) = 1,$$
$$\lim_{n\to\infty} M(STx_n, Tx, t) = 1, \quad \lim_{n\to\infty} M(TTx_n, TSx_n, t) = 1,$$
$$\lim_{n\to\infty} M(TTx_n, Sx, t) = 1, \quad \lim_{n\to\infty} M(TSx_n, Sx, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $x \in X$ and $t > 0$.

Definition 4.12 [32]: [M. S. Khan et al., 2007]

Two self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **compatible of type (A-1)** if

$$\lim_{n\to\infty} M(STx_n, TTx_n, t) = 1 \quad \text{for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x \quad \text{for some } x \in X.$$

Definition 4.13 [32]:

Two self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **compatible of type (A-2)** if

$$\lim_{n\to\infty} M(TSx_n, SSx_n, t) = 1 \quad \text{for all } t > 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x \quad \text{for some } x \in X.$$

for some $x \in X$ and $t > 0$.

Remark 4.14: If a pair of mappings (S, T) is compatible of type (A-1), then the pair (T, S) is compatible of type (A-2). Further, if S and T are compatible mappings of type (A), then the pair (S, T) is compatible of both type (A-1) and type (A-2).

Definition 4.15 [47]: [Y. Rohen et al., 2008]

The self-mappings S and T of a fuzzy metric space $(X, M, *)$ are said to be **compatible of type (R)** if

$$\lim_{n\to\infty} M(STx_n, TSx_n, t) = 1 \quad \text{and} \quad \lim_{n\to\infty} M(SSx_n, TTx_n, t) = 1,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = x$$

for some $x \in X$ and $t > 0$.

Remark 4.16:

The compatible mappings of type (K) in a fuzzy metric space are independent of compatible mappings, compatible mappings of type (A), and compatible mappings of type (P).

Example 4.17 [20]:

Let $X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$. Define

$$M(x, y, t) = \frac{t}{t + d(x, y)}, \quad \forall x, y \in X, t > 0,$$

and

$$a * b = ab, \quad \forall a, b \in [0, 1].$$

Then $(X, M, *)$ is a fuzzy metric space. Define self-mappings S and T as:

$$Sx = \begin{cases} 2, & x \in [0,1) - \left\{\frac{1}{2}\right\}, \\ 0, & x = \frac{1}{2}, \\ \frac{2-x}{2}, & x \in (1, 2], \end{cases} \qquad Tx = \begin{cases} 0, & x \in [0,1) - \left\{\frac{1}{2}\right\}, \\ 2, & x = \frac{1}{2}, \\ \frac{x}{2}, & x \in (1, 2]. \end{cases}$$

S and T are not continuous at $x = 1, \frac{1}{2}$. Consider a sequence $\{x_n\} \subset X$ given by

$$x_n = 1 + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Then S and T are compatible of type (K), but they are neither compatible nor compatible of type (A) [compatible of type (P)].

Let us consider another example in the same space $X = [0, 2]$ with the same metric and fuzzy metric structure. Define self-mappings S and T as:

$$Sx = Tx = \begin{cases} 1, & x \in (0, 1], \\ \frac{4}{3}, & x = 1, \\ 2 - x, & x \in (1, 2], \end{cases} \qquad Tx = \begin{cases} 1, & x \in (0, 1], \\ \frac{4}{3}, & x = 1, \\ x, & x \in (1, 2]. \end{cases}$$

Consider the sequence $\{x_n\} \subset X$ given by

$$x_n = 1 + \frac{1}{n}, \quad n \in \mathbb{N}.$$

Then S and T are not compatible of type (K), but they are compatible, compatible of type (A), and compatible of type (P).

5. Result and Conclusion

5.1. Interrelationship of Weaker Forms of Commuting Mappings.

(1) Every commuting mapping $\implies$ weakly commuting, but the converse is not true.
(2) Compatible mappings of type (A) $\implies$ Compatible mappings of type (B) $\implies$ Compatible mappings of type (C), but the converse is not true in general.
(3) Compatible of type (E) implies both S- and T-compatible of type (E); however, S- or T-compatible of type (E) do not imply compatible of type (E).
(4) If $St = Tt$, then compatible mappings of type (E) imply compatible mappings of type (A), compatible mappings of type (B), compatible mappings of type (C), and compatible mappings of type (P); however, the converse may not be true.
(5) If $St \neq Tt$, then compatible mappings of type (E) are neither compatible mappings nor compatible mappings of type (A), type (C), or type (P).
(6) Weak commutativity in a fuzzy metric space implies R-weak commutativity, and the converse is true for $R \leq 1$.
(7) Two self-mappings S and T of a fuzzy metric space $(X, M, *)$: if S and T are R-weakly commuting, then S and T are semi-compatible; however, the converse is not true.
(8) Two self-mappings S and T of a fuzzy metric space $(X, M, *)$: if S and T are semi-compatible, then S and T are compatible; however, the converse is not true.
(9) The compatible mappings of type (K) in a fuzzy metric space are independent of compatible mappings, compatible mappings of type (A), and compatible mappings of type (P).

5.2. **Conclusion.** Some compatible mappings in metric space and compatible mappings in fuzzy metric space have been pointed out in addition to various compatibilities in metric spaces and fuzzy metric spaces. This helps in the comparative and relational study in these spaces for researchers.

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Nandu Prasad Koiri
Department of Mathematics, Tri-Chandra Multiple Campus,Tribhuvan University, Kathmandu, Nepal, PhD Scholar of Arunodaya University, Arunachal, India
E-mail address: kushwahanandu072@gmail.com

Rajeev Kumar
Department of Mathematics, Arunodaya University, Itanagar, Arunachal, India
E-mail address: rjv2878@gmail.com

K. B. Manandhar
Department of Mathematics, School of Science, Kathmandu University, Nepal
E-mail address: kb.manandhar@ku.edu.np

Ajay Kumar Chaudhary, *corresponding author
Department of Mathematics, Tri-Chandra Multiple Campus,Tribhuvan University, Kathmandu, Nepal
E-mail address: akcsaurya81@gmail.com